

Tetravalent 2-arc-transitive Cayley graphs on non-abelian simple groups

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Abstract

A graph Γ is said to be *2-arc-transitive* if its full automorphism group $\text{Aut}(\Gamma)$ has a single orbit on ordered paths of length 2, and for $G \leq \text{Aut}(\Gamma)$, Γ is *G-regular* if G is regular on the vertex set of Γ . Let G be a finite non-abelian simple group and let Γ be a connected tetravalent 2-arc-transitive G -regular graph. In 2004, Fang, Li and Xu proved that either $G \trianglelefteq \text{Aut}(\Gamma)$ or G is one of 22 possible candidates. In this paper, the number of candidates is reduced to 7, and for each candidate G , it is shown that $\text{Aut}(\Gamma)$ has a normal arc-transitive non-abelian simple subgroup T such that $G \leq T$ and the pair (G, T) is explicitly given.

Keywords: Cayley graph, coset graph, simple group.

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1 Introduction

Throughout this paper, all groups and graphs are finite, and all graphs are simple and undirected. Let G be a permutation group on a set Ω and let $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G , that is, the subgroup of G fixing the point α . The group G is *semiregular* if $G_\alpha = 1$ for every $\alpha \in \Omega$, and *regular* if G is transitive and semiregular.

For a graph Γ , denote by $V(\Gamma)$, $E(\Gamma)$ and $\text{Aut}(\Gamma)$ its vertex set, edge set and full automorphism group, respectively. An s -arc in Γ is an ordered $(s+1)$ -tuple (v_0, v_1, \dots, v_s) of vertices of Γ such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$. Let $G \leq \text{Aut}(\Gamma)$. The graph Γ is said to be (G, s) -arc-transitive or G -regular if G has a single orbit on s -arcs of Γ or is regular on vertices of Γ , respectively. For short, a 1-arc means an *arc*, and $(G, 1)$ -arc-transitive means G -arc-transitive. The graph Γ is said to be s -arc-transitive if it is $(\text{Aut}(\Gamma), s)$ -arc-transitive. In particular, 0-arc-transitive is *vertex-transitive*, and 1-arc-transitive is *arc-transitive* or *symmetric*.

For a group G and a subset S of G such that $1 \notin S$ and $S^{-1} = S$, the *Cayley graph* $\text{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} | g \in G, s \in S\}$. For $g \in G$, the map $R(g) : x \mapsto xg$ for $x \in G$ is a permutation

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on G , and $R(G) = \{R(g) \mid g \in G\}$ consists of a permutation group on G , called the right regular representation of G . It is easy to see that $R(G) \leq \text{Aut}(\Gamma)$. A Cayley graph $\Gamma = \text{Cay}(G, S)$ is said to be *normal* if $R(G)$ is a normal subgroup of $\text{Aut}(\Gamma)$, and in this case, $\text{Aut}(\Gamma) = R(G) \rtimes \text{Aut}(G, S)$ by Godsil [6] or Xu [20], where $\text{Aut}(G, S) = \{\alpha \in \text{Aut}(G) \mid S^\alpha = S\}$. Note that $\text{Cay}(G, S)$ is $R(G)$ -regular. It is well-known that if Γ is G -regular then Γ is isomorphic to a Cayley graph on G .

The investigation of symmetric graphs has a long and interesting history, highlighted at an early stage by ingenious work by Tutte [23, 24] on the cubic case. Let Γ be a connected symmetric cubic Cayley graph on a non-abelian simple group G . Li [10] proved that either Γ is normal or $G = A_5, A_7, \text{PSL}(2, 11), M_{11}, A_{11}, A_{15}, M_{23}, A_{23}$ or A_{47} . Based on Li's work, Xu *et al* [21, 22] proved that either Γ is normal or $G = A_{47}$, and for the latter, Γ is 5-arc-transitive and there are only two such graphs.

Let Γ be a connected tetravalent 2-arc-transitive Cayley graph on a non-abelian simple group G . Fang *et al* [3] proved that either Γ is normal, or G is one of the 22 possible candidates listed in [3, Table 1] (the group A_{23} is missed in the table). The following is the main result of this paper.

Theorem 1.1 *Let G be a non-abelian simple group and Γ a connected tetravalent 2-arc-transitive G -regular graph. Then either $G \trianglelefteq \text{Aut}(\Gamma)$ or $\text{Aut}(\Gamma)$ contains a normal arc-transitive non-abelian simple subgroup T such that $G \leq T$ and (G, T) is listed in Table 1.*

G	M_{11}	$A_{2^3.3-1}$	$A_{2^2.3^2-1}$	$A_{2^3.3^2-1}$	$A_{2^4.3^2-1}$	$A_{2^4.3^3-1}$	$A_{2^4.3^6-1}$
T	M_{12}	$A_{2^3.3}$	$A_{2^2.3^2}$	$A_{2^3.3^2}$	$A_{2^4.3^2}$	$A_{2^4.3^3}$	$A_{2^4.3^6}$

Table 1: 7 possible pairs of non-abelian simple groups

For connected symmetric cubic Cayley graphs on non-abelian simple groups, similar to Theorem 1.1, there are six possible pairs $(G, T) = (A_{47}, A_{48}), (\text{PSL}(2, 11), M_{11}), (M_{11}, M_{12}), (A_{11}, A_{12}), (M_{23}, M_{24})$ or (A_{23}, A_{24}) (see [10, Theorem 7.1.3]), and Xu *et al* [21, 22] proved that only the pair $(G, T) = (A_{47}, A_{48})$ can happen and there are exactly two connected non-normal symmetric cubic Cayley graphs on A_{47} . For the 7 possible pairs of (G, T) in Theorem 1.1, by MAGMA [1] there is only one 2-arc-transitive Cayley graph on M_{11} for $(G, T) = (M_{11}, M_{12})$ (see Remark of Lemma 3.2), and there are four 2-arc-transitive Cayley graphs on A_{23} for $(G, T) = (A_{23}, A_{24})$. The number of 3-arc-transitive Cayley graphs on A_{35} for $(G, T) = (A_{35}, A_{36})$ is 4, on A_{71} for $(G, T) = (A_{71}, A_{72})$ is 18, and on A_{143} for $(G, T) = (A_{143}, A_{144})$ is 31. At the end of this paper, we give examples of connected non-normal 2-arc-transitive Cayley graph on A_{23} for $(G, T) = (A_{23}, A_{24})$ (see Example 3.4).

2 Preliminaries

In this section, we describe some preliminary results which will be used later. The first result is the stabilizers of tetravalent 2-arc-transitive graphs, given in [14, Theorem 4].

Proposition 2.1 *Let Γ be a connected tetravalent (G, s) -arc-transitive but not $(G, s+1)$ -arc-transitive graph with $v \in V(\Gamma)$. If $s \geq 2$ then one of the following occurs:*

- (1) *For $s = 2$, we have $G_v \cong A_4$ or S_4 . In particular, $|G_v| = 2^2 \cdot 3$ or $2^3 \cdot 3$.*
- (2) *For $s = 3$, we have $G_v \cong \mathbb{Z}_3 \times A_4$, $\mathbb{Z}_3 \rtimes S_4$, or $S_3 \times S_4$. In particular, $|G_v| = 2^2 \cdot 3^2$, $2^3 \cdot 3^2$ or $2^4 \cdot 3^2$.*
- (3) *For $s = 4$, we have $G_v \cong \mathbb{Z}_2^3 \rtimes \text{GL}(2, 3) = \text{AGL}(2, 3)$. In particular, $|G_v| = 2^4 \cdot 3^3$.*
- (4) *For $s = 7$, we have $G_v \cong \mathbb{Z}_3^5 \rtimes \text{GL}(2, 3)$. In particular, $|G_v| = 2^4 \cdot 3^6$.*

Remark: For $s = 7$, by [14, Theorem 1.1], $G_v = \langle e_0, e_1, e_2, e_3, e_4, e_5, e_6, d \rangle \cong \mathbb{Z}_3^5 \rtimes \text{GL}(2, 3)$ with the following relations: $e_i^3 = d^2 = 1$ for all $0 \leq i \leq 6$, $[e_i, e_j] = 1$ for all $|i - j| < 4$, $[e_0, e_4] = e_2$, $[e_0, e_5] = e_1^{-1}e_2e_3^{-1}e_4^{-1}$, $e_0e_6e_0 = e_6e_0e_6$, $[e_1, e_5] = e_3$, $e_1^{e_6} = e_5e_4e_3^{-1}e_2e_1$, $e_2^{e_6} = e_2e_4^{-1}$, $e_k^d = e_k^{-1}$ for $k = 0, 2, 3, 5, 6$ and $e_l^d = e_l$ for $l = 1, 4$, and by MAGMA [1], G_v has no normal subgroup isomorphic to A_4 , S_4 , $\mathbb{Z}_3 \times A_4$, $\mathbb{Z}_3 \rtimes S_4$, $S_3 \times S_4$, $\text{AGL}(2, 3)$, or a non-trivial 2-group.

Let Γ be a graph and let $N \leq \text{Aut}(\Gamma)$. The *quotient graph* Γ_N of Γ relative to N is defined as the graph with vertices the orbits of N on $V(\Gamma)$ and with two orbits adjacent if there is an edge in Γ between those two orbits. The theory of quotient graph is widely used to investigate symmetric graphs. The following proposition can be deduced from [5, Theorem 1.1] and [15, Theorem 4.1].

Proposition 2.2 *Let Γ be a connected tetravalent (G, s) -arc-transitive graph for some $s \geq 2$ and let N be a normal subgroup of G . If N is transitive then either N is regular or arc-transitive, and if N has at least three orbits then N acts semiregularly on $V(\Gamma)$ and the quotient graph Γ_N is a connected tetravalent $(G/N, s)$ -arc-transitive graph.*

Guralnick [7] classified non-abelian simple groups which contain subgroups of index a power of a prime, and by [11, Theorem 1], we have the following result.

Proposition 2.3 *Let G and T be non-abelian simple groups such that $G \leq T$ and $|T : G| = 2^a \cdot 3^b \geq 6$ with $0 \leq a \leq 4$ and $0 \leq b \leq 6$. Then T , G and $|T : G|$ are listed in Table 2.*

T	G	$ T : G $	T	G	$ T : G $
M_{11}	$\text{PSL}(2, 11)$	$2^2 \cdot 3$	M_{12}	M_{11}	$2^2 \cdot 3$
M_{24}	M_{23}	$2^3 \cdot 3$	$\text{PSU}(3, 3)$	$\text{PSL}(2, 7)$	$2^2 \cdot 3^2$
A_9	A_7	$2^3 \cdot 3^2$	$\text{PSp}(4, 3)$	A_6	$2^3 \cdot 3^2$
$\text{PSp}(6, 2)$	A_8	$2^3 \cdot 3^2$	$\text{PSU}(4, 3)$	$\text{PSL}(3, 4)$	$2 \cdot 3^4$
M_{12}	$\text{PSL}(2, 11)$	$2^4 \cdot 3^2$	$\text{PSU}(4, 3)$	A_7	$2^4 \cdot 3^4$
$G_2(3)$	$\text{PSL}(2, 13)$	$2^4 \cdot 3^5$	A_n	A_{n-1}	$n = 2^a \cdot 3^b$

Table 2: Non-abelian simple group pairs of index $2^a \cdot 3^b$

Let G and E be two groups. We call an extension E of G by N a *central extension* of G if E has a central subgroup N such that $E/N \cong G$, and if further E is perfect, that is, the derived group $E' = E$, we call E a *covering group* of G . A covering group E of G is called a *double cover* if $|E| = 2|G|$. Schur [17] proved that for every non-abelian simple group G there is a unique maximal covering group M such that every covering group of G is a factor group of M (see [8, Kapitel V, §23]). This group M is called the *full covering group* of G , and the center of M is the *Schur multiplier* of G , denoted by $\text{Mult}(G)$.

Proposition 2.4 *For $n \geq 5$, the alternating group A_n has a unique double cover $2.A_n$, and for $n \geq 7$, all subgroups of index n of $2.A_n$ are conjugate and isomorphic to $2.A_{n-1}$.*

Proof: By Kleidman and Liebeck [9, Theorem 5.1.4], $\text{Mult}(A_n) \cong \mathbb{Z}_2$ for $n \geq 5$ with $n \neq 6, 7$, and $\text{Mult}(A_n) \cong \mathbb{Z}_6$ for $n = 6$ or 7 . This implies that A_n has a unique double cover for $n \geq 5$, and we denote it by $2.A_n$. Since A_n has no proper subgroup of index less than n , all subgroups of index n of $2.A_n$ contain the center of order 2 of $2.A_n$. Let $n \geq 7$. By [19, 2.7.2], $2.A_n$ contains a subgroup $2.A_{n-1}$ of index n , and since all subgroups of index n of A_n are conjugate, all subgroups of index n of $2.A_n$ are conjugate and hence isomorphic to $2.A_{n-1}$. \square

Now, we introduce the so called coset graph. Let G be a group and $H \leq G$. Denote by D a union of some double cosets of H in G such that $D^{-1} = D$. The *coset graph* $\Gamma = \text{Cos}(G, H, D)$ on G with respect to H and D is defined to have vertex set $V(\Gamma) = [G : H]$, the set of right cosets of H in G , and the edge set $E(\Gamma) = \{\{Hg, Hd\} \mid g \in G, d \in D\}$. It is well known that $\Gamma = \text{Cos}(G, H, D)$ has valency $|D|/|H|$ and it is connected if and only if $G = \langle D, H \rangle$, that is, D and H generate G . The action of G on $[G : H]$ by right multiplication induces a vertex-transitive group of automorphisms of Γ , and this group is arc-transitive if and only if D is a single double coset. Moreover, this action is faithful if and only if $H_G = 1$, where H_G is the largest normal subgroup of G contained in H . Clearly, $\text{Cos}(G, H, D) \cong \text{Cos}(G, H^\alpha, D^\alpha)$ for each $\alpha \in \text{Aut}(G)$.

Conversely, let Γ be a G -vertex-transitive graph. By [16], Γ is isomorphic to a coset graph $\text{Cos}(G, H, D)$, where $H = G_v$ is the vertex stabilizer of $v \in V(\Gamma)$ in G and D consists of all elements of G which map v to one of its neighbors. It is easy to show that $H_G = 1$ and D is a union of some double cosets of H in G satisfying $D^{-1} = D$. Assume that G is arc-transitive and $g \in G$ interchanges v and one of its neighbors. Then $g^2 \in H$ and $D = HgH$. Furthermore, g can be chosen as a 2-element in G , and the valency of Γ is $|D|/|H| = |H : H \cap H^g|$. For more details regarding coset graph, refer to [4, 12, 13, 16].

Proposition 2.5 *Let Γ be a connected G -arc-transitive graph and $\{u, v\}$ an edge of Γ . Then Γ is isomorphic to a coset graph $\text{Cos}(G, G_v, G_v g G_v)$, where g is a 2-element in G such that $G_{uv}^g = G_{uv}$, $g^2 \in G_v$ and $\langle G_v, g \rangle = G$. Moreover, Γ has valency $|G_v : G_v \cap G_v^g|$.*

In Proposition 2.5, the 2-element g is called *feasible* to G and G_v . Feasible g can be found by MAGMA [1] when the order $|G|$ is not too large, and for convenience, an example of computer program to find feasible g is given as an appendix at the end of the paper, which will be used in Section 3 frequently.

3 Proof of Theorem 1.1

In this section, we always assume that G is a non-abelian simple group. To prove Theorem 1.1, we need the following three lemmas.

Lemma 3.1 *There is no connected tetravalent A_{12} -arc-transitive A_{11} -regular graph, and no connected tetravalent M_{12} -arc-transitive graph with stabilizer isomorphic to S_4 .*

Proof: Let Γ be a connected tetravalent T -arc-transitive graph with $v \in V(\Gamma)$. By Proposition 2.5, $\Gamma = \text{Cos}(T, T_v, T_v t T_v)$ for some feasible t , that is, t is a 2-element such that $t^2 \in T_v$, $\langle T_v, t \rangle = T$ and $|T_v : T_v \cap T_v^t| = 4$.

Let $T = A_{12}$ and let Γ be A_{11} -regular. Then $|A_{11}| = |V(\Gamma)|$ and $|T_v| = |T|/|A_{11}| = 12$. By Proposition 2.1, $T_v \cong A_4$. By MAGMA [1], T_v has 12 conjugacy classes in T . Take a given T_v in each conjugacy class, and computation shows that there is no such feasible t . It follows that there is no connected tetravalent A_{12} -arc-transitive A_{11} -regular graph.

Let $T = M_{12}$ and $T_v \cong S_4$. By MAGMA, T_v has four conjugacy classes in T . Take a given T_v in each conjugacy class, and computation shows that there is no such feasible t . Thus, there is no connected tetravalent M_{12} -arc-transitive graph with stabilizer isomorphic to S_4 . \square

The *radical* of a group is defined as its largest soluble normal subgroup.

Lemma 3.2 *Let Γ be a connected tetravalent G -vertex-transitive graph and let X be a 2-arc-transitive subgroup of $\text{Aut}(\Gamma)$ containing G . If X has trivial radical, then either $G \trianglelefteq X$, or X has a normal arc-transitive non-abelian simple subgroup T such that $G \leq T$ and $(G, T) = (M_{11}, M_{12})$ or (A_{n-1}, A_n) with $n \geq 8$ and $n \mid 2^4 \cdot 3^6$. Moreover, if Γ is G -regular then (G, T) is listed in Table 1.*

Proof: Let N be a minimal normal subgroup of X . Since X has trivial radical, $N = T^s$ for a positive integer s and a non-abelian simple group T . Clearly, $NG \leq X$. Since Γ is G -vertex-transitive, by the Frattini argument we have $X = GX_v$ for $v \in V(\Gamma)$, and hence $|X| = |G||X_v|/|G_v|$. Since $|NG| = |N||G|/|N \cap G|$, we have $|N|/|N \cap G| \mid |X_v|/|G_v|$, and by Proposition 2.1, $|N|/|N \cap G| \mid 2^4 \cdot 3^6$.

The simplicity of G implies that $N \cap G = 1$ or G . If $N \cap G = 1$ then $|N| = |N|/|N \cap G| \mid 2^4 \cdot 3^6$, which is impossible because N is insoluble. Thus, $N \cap G = G$, that is, $G \leq N$, and $|N|/|G| \mid 2^4 \cdot 3^6$. Since $T \trianglelefteq N$, we have $T \cap G = 1$ or G . If $T \cap G = 1$ then $|T| = |T|/|G \cap T| = |GT|/|G| \mid |N|/|G|$, which also is impossible because $|N|/|G| \mid 2^4 \cdot 3^6$. Thus, $G \cap T = G$, that is, $G \leq T$. Since $|N|/|G| = |T|^{s-1}|T : G| \mid 2^4 \cdot 3^6$, we have $s = 1$. It follows that $G \leq T \trianglelefteq X$ and $|T : G| = |T|/|G| \mid 2^4 \cdot 3^6$.

If $G = T$ then $G \trianglelefteq X$ and we are done. Now we may assume that G is a proper subgroup of T . In particular, $G \not\trianglelefteq X$. By Proposition 2.3, (G, T) is listed in Table 2. Since $G \leq T \triangleleft X$, Γ is T -vertex-transitive, and Proposition 2.2 implies that Γ is T -arc-transitive. By Proposition 2.5, $\Gamma = \text{Cos}(T, T_v, T_v t T_v)$ for some feasible t . Note that $|T|/|G| = |V(\Gamma)||T_v|/(|V(\Gamma)||G_v|) = |T_v|/|G_v|$.

Suppose $(G, T) = (\text{PSL}(2, 13), G_2(3))$. Then $|T_v|/|G_v| = |T|/|G| = 2^4 \cdot 3^5 \mid |T_v|$. By Proposition 2.1, $T_v \cong \mathbb{Z}_3^5 \rtimes \text{GL}(2, 3)$, and $|V(\Gamma)| = |T|/|T_v| = 364$. However, there is no connected tetravalent 2-arc-transitive graph of order 364 by [14, Table 2], a contradiction.

Suppose $(G, T) = (\text{PSL}(3, 4), \text{PSU}(4, 3))$ or $(A_7, \text{PSU}(4, 3))$. Then $3^4 \mid |T_v|$ because $|T_v|/|G_v| = |T|/|G|$. By Proposition 2.1, $T_v \cong \mathbb{Z}_3^5 \rtimes \text{GL}(2, 3)$, and by MAGMA, $\text{PSU}(4, 3)$ has no subgroup isomorphic to $\mathbb{Z}_3^5 \rtimes \text{GL}(2, 3)$, a contradiction.

Suppose $(G, T) = (A_6, \text{PSp}(4, 3))$. Then $|T_v|/|G_v| = |T|/|G| = 2^3 \cdot 3^2 \mid |T_v|$. Since $|T| = 2^6 \cdot 3^4 \cdot 5$, Proposition 2.1 implies that $T_v \cong \mathbb{Z}_3 \rtimes S_4$, $S_3 \times S_4$ or $\text{AGL}(2, 3)$, and by MAGMA, $\text{PSp}(4, 3)$ has no subgroup isomorphic to $\mathbb{Z}_3 \rtimes S_4$, $S_3 \times S_4$ or $\text{AGL}(2, 3)$, a contradiction.

Suppose $(G, T) = (\text{PSL}(2, 7), \text{PSU}(3, 3))$. Then $|T_v|/|G_v| = |T|/|G| = 2^2 \cdot 3^2 \mid |T_v|$. Since $|T| = 2^5 \cdot 3^2 \cdot 7$, Proposition 2.1 implies that $T_v \cong \mathbb{Z}_3 \times A_4$, $\mathbb{Z}_3 \rtimes S_4$ or $S_3 \times S_4$, and by MAGMA, $\text{PSU}(3, 3)$ has no subgroup isomorphic to $\mathbb{Z}_3 \times A_4$, $\mathbb{Z}_3 \rtimes S_4$ or $S_3 \times S_4$, a contradiction.

Suppose $(G, T) = (\text{PSL}(2, 11), M_{11})$. Then $|T_v|/|G_v| = |T|/|G| = 2^2 \cdot 3 \mid |T_v|$. Since $|T| = 2^4 \cdot 3^2 \cdot 5 \cdot 7$, Proposition 2.1 implies that $T_v \cong A_4$, S_4 , $\mathbb{Z}_3 \times A_4$, $\mathbb{Z}_3 \rtimes S_4$ or $S_3 \times S_4$. By MAGMA, M_{11} has no subgroup isomorphic to $\mathbb{Z}_3 \times A_4$, $\mathbb{Z}_3 \rtimes S_4$ or $S_3 \times S_4$, and if $T_v \cong A_4$ or S_4 then T_v has one conjugacy class, respectively. By taking a given T_v in each conjugacy class, computation shows that there is no feasible t , a contradiction.

Suppose $(G, T) = (A_7, A_9)$. Then $|T_v|/|G_v| = |T|/|G| = 2^3 \cdot 3^2 \mid |T_v|$. Since $3^5 \nmid |A_9|$, Proposition 2.1 implies that $T_v \cong \mathbb{Z}_3 \rtimes S_4$, $S_3 \times S_4$ or $\text{AGL}(2, 3)$. By MAGMA, A_9 has no subgroup isomorphic to $\text{AGL}(2, 3)$ and if $T_v \cong \mathbb{Z}_3 \rtimes S_4$ or $S_3 \times S_4$ then T_v has two conjugacy classes, respectively. By taking a given T_v in each conjugacy class, computation shows that there is no feasible t , a contradiction.

Suppose $(G, T) = (M_{23}, M_{24})$. Then $|T_v|/|G_v| = |T|/|G| = 2^3 \cdot 3$ and $2^3 \cdot 3 \mid |T_v|$. Since $3^4 \nmid |M_{24}|$, Proposition 2.1 implies that $T_v \cong S_4$, $\mathbb{Z}_3 \rtimes S_4$, $S_3 \times S_4$ or $\text{AGL}(2, 3)$. By MAGMA, if $T_v \cong S_4$ then T_v has 19 conjugacy classes, if $T_v \cong \text{AGL}(2, 3)$ then T_v has one conjugacy class, and if $T_v \cong \mathbb{Z}_3 \rtimes S_4$ or $S_3 \times S_4$ then T_v has four conjugacy classes, respectively. By taking a given T_v in each conjugacy class, computation shows that there is no feasible t , a contradiction.

Suppose $(G, T) = (A_8, \text{PSp}(6, 2))$. Then $|T_v|/|G_v| = |T|/|G| = 2^3 \cdot 3^2 \mid |T_v|$. Since $3^5 \nmid |\text{PSp}(6, 2)|$, Proposition 2.1 implies $T_v \cong \mathbb{Z}_3 \rtimes S_4$, $S_3 \times S_4$ or $\text{AGL}(2, 3)$. By MAGMA, $\text{PSp}(6, 2)$ has no subgroup isomorphic to $\text{AGL}(2, 3)$. If $T_v \cong \mathbb{Z}_3 \rtimes S_4$ then T_v has four conjugacy classes, and if $T_v \cong S_3 \times S_4$ then T_v has eight conjugacy classes. By taking a given T_v in each conjugacy class, computation shows that there is no feasible t , a contradiction.

Suppose $(G, T) = (\text{PSL}(2, 11), M_{12})$. Then $|T_v|/|G_v| = |T|/|G| = 2^4 \cdot 3^2 \mid |T_v|$. Since $|T| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$, Proposition 2.1 implies that $T_v \cong S_3 \times S_4$ or $\text{AGL}(2, 3)$. By MAGMA, M_{12} has no subgroup isomorphic to $S_3 \times S_4$ and if $T_v \cong \text{AGL}(2, 3)$ then T_v has two conjugacy classes. By taking a given T_v in each conjugacy class, computation shows that there is no feasible t , a contradiction.

Suppose $(G, T) = (A_5, A_6)$. Then $|T_v|/|G_v| = |T|/|G| = 2 \cdot 3 \mid |T_v|$. By Atlas [2, pp.4] and Proposition 2.1, $T_v \cong A_4$ or S_4 , and by MAGMA, T_v has two conjugacy classes, respectively. By taking a given T_v in each conjugacy class, computation shows that there is no feasible t , a contradiction.

By the above contradictions, $(G, T) \neq (\text{PSL}(2, 13), G_2(3)), (\text{PSL}(3, 4), \text{PSU}(4, 3)), (A_7, \text{PSU}(4, 3)), (A_6, (\text{PSp}(4, 3))), (\text{PSL}(2, 7), \text{PSU}(3, 3)), (\text{PSL}(2, 11), M_{11}), (A_7, A_9), (M_{23}, M_{24}), (A_8, \text{PSp}(6, 2)), (\text{PSL}(2, 11), M_{12})$ or (A_5, A_6) . Deleting the above impossible pairs from Table 2, we have $(G, T) = (M_{11}, M_{12})$ or (A_{n-1}, A_n) with $n \geq 8$ and $n \mid 2^4 \cdot 3^6$ because $|T|/|G| \mid 2^4 \cdot 3^6$. To finish the proof, let Γ be G -regular. Then $G_v = 1$ and $|T_v| = |T_v|/|G_v| = |T|/|G|$. By Proposition 2.1, $(G, T) = (M_{11}, M_{12})$ or (A_{n-1}, A_n) with $n = 2^2 \cdot 3, 2^3 \cdot 3, 2^2 \cdot 3^2, 2^3 \cdot 3^2, 2^4 \cdot 3^2, 2^4 \cdot 3^3$ or $2^4 \cdot 3^6$, and by Lemma 3.1, $(G, T) \neq (A_{11}, A_{12})$. It follows that (G, T) is listed in Table 1. \square

Remark: Let $(G, T) = (M_{11}, M_{12})$. Then there is a unique connected tetravalent T -arc-transitive G -regular graph Γ , and $\text{Aut}(\Gamma) \cong M_{12} \rtimes \mathbb{Z}_6$ has non-trivial radical \mathbb{Z}_3 . These facts can be checked by MAGMA. In fact, since $|T_v| = |T_v|/|G_v| = |T|/|G| = 2^2 \cdot 3$, Proposition 2.1 implies that $T_v \cong A_4$. By Proposition 2.5, $\Gamma = \text{Cos}(T, T_v, T_v t T_v)$ for some feasible t , and by MAGMA, computation shows that T_v has four conjugacy classes in T . Take a given T_v in each conjugacy class: for two conjugacy classes there is no feasible t , and for the other two conjugacy classes, one has 24 feasible t but all corresponding graphs Γ are not M_{11} -vertex-transitive, and the other has 12 feasible t , of which the corresponding graphs Γ are isomorphic to each other and $\text{Aut}(\Gamma) = M_{12} \rtimes \mathbb{Z}_6$ with radical \mathbb{Z}_3 . \square

Lemma 3.3 *Let Γ be a connected tetravalent 2-arc-transitive G -regular graph and let $\text{Aut}(\Gamma)$ have non-trivial radical R with at least three orbits on $V(\Gamma)$. Assume $RG = R \times G$. Then $G \trianglelefteq \text{Aut}(\Gamma)$ or $\text{Aut}(\Gamma)$ contains a normal arc-transitive non-abelian simple subgroup T such that $G \leq T$ and (G, T) is listed in Table 1.*

Proof: Set $A = \text{Aut}(\Gamma)$ and $B = RG = R \times G$. Then G is characteristic in B . To finish the proof, we may assume $G \not\trianglelefteq A$ and aim to show that A contains a normal arc-transitive non-abelian simple subgroup T such that $G \leq T$ and (G, T) is listed in Table 1.

Since $R \neq 1$ has at least three orbits, by Proposition 2.2 the quotient graph Γ_R is a connected tetravalent $(A/R, 2)$ -arc-transitive graph with $A/R \leq \text{Aut}(\Gamma_R)$, and Γ_R is B/R -vertex-transitive. Since $G \not\trianglelefteq A$ and G is characteristic in B , we have $B \not\trianglelefteq A$ and $G \cong B/R \not\trianglelefteq A/R$. Furthermore, A/R has trivial radical as R is the radical of A . By Lemma 3.2, A/R has a normal arc-transitive subgroup I/R such that $B/R \leq I/R$ and $(B/R, I/R) \cong (G, T) = (M_{11}, M_{12})$ or (A_{n-1}, A_n) with $n \geq 8$ and $n \mid 2^4 \cdot 3^6$.

Note that $I \trianglelefteq A$. Since A is 2-arc-transitive, I is arc-transitive. Let $C = C_I(R)$. Then $C \trianglelefteq I$ and $C \cap R \leq Z(C)$. Since $B = R \times G \leq I$, we have $G \leq C$, and since $C/C \cap R \cong CR/R \trianglelefteq I/R \cong T$, we have $C \cap R = Z(C)$, $C/Z(C) \cong T$ and $I = CR$. Furthermore, $C'/C' \cap Z(C) \cong C'Z(C)/Z(C) = (C/Z(C))' = C/Z(C) \cong T$. Thus, $Z(C') = C' \cap Z(C)$, $C = C'Z(C)$ and $C'/Z(C') \cong T$. It follows that $C' = (C'Z(C))' = C''$, and so C' is a covering group of T . Since C/C' is abelian, $G \leq C'$.

Recall that $(G, T) = (M_{11}, M_{12})$ or (A_{n-1}, A_n) with $n \geq 8$ and $n \mid 2^4 \cdot 3^6$. By [9, Theorem 5.1.4] and [2, pp.31], the Schur multiplier $\text{Mult}(A_n) = \mathbb{Z}_2$ for $n \geq 8$ and $\text{Mult}(M_{12}) = \mathbb{Z}_2$. Then T has a unique double cover, denoted by $2.T$.

Suppose $Z(C') = \mathbb{Z}_2$. Then C' is the unique double cover of T , that is, $C' = 2.T$. Let $(G, T) = (M_{11}, M_{12})$. Then $|C'| = 2|M_{12}|$, and since Γ is G -regular, $|C'_v| = 24$ for

$v \in V(\Gamma)$. In particular, $3 \mid |I_v|$, and I is 2-arc-transitive by the arc-transitivity of I . Since $C \trianglelefteq I$ and C' is characteristic in C , we have $C' \trianglelefteq I$ and so C' is arc-transitive. Similarly, C' is 2-arc-transitive as $3 \mid |C'_v|$, and by Proposition 2.1, $C'_v \cong S_4$. The quotient graph $\Gamma_{Z(C')}$ is a connected tetravalent $(C'/Z(C'), 2)$ -arc-transitive graph with stabilizer isomorphic to C'_v . Since $C'_v \cong S_4$ and $C'/Z(C') \cong M_{12}$, this is impossible by Lemma 3.1. Now let $(G, T) = (A_{n-1}, A_n)$ with $n \geq 8$. Then $C' = 2.A_n$. Since $Z(C') \trianglelefteq I$ and $I/R \cong T$, we have $Z(C') \leq R$, and since $B = G \times R$, we have $G \times Z(C') \leq C'$. Then $G \times Z(C')$ is a subgroup of index n of C' isomorphic to $A_{n-1} \times \mathbb{Z}_2$. This is impossible by Proposition 2.4.

Thus, $Z(C') = 1$. It follows that $C' \cong T$ and $G \leq C' \trianglelefteq I$. Since $|I| = |I/R||R| = |T||R| = |C'||R|$ and $C' \cap R = 1$, we have $I = C' \times R$ because $C' \trianglelefteq I$, and so C' is characteristic in I . Since $I \trianglelefteq A$ and A is 2-arc-transitive, A has a normal arc-transitive non-abelian simple subgroup $C' \cong T$ containing G and $(G, T) = (M_{11}, M_{12})$ or (A_{n-1}, A_n) with $n \geq 8$ and $n \mid 2^4 \cdot 3^6$. Since Γ is G -regular, $|C'_v| = |C'_v|/|G_v| = |C'|/|G| = |T : G|$, and by Proposition 2.1, $(G, T) = (M_{11}, M_{12})$ or (A_{n-1}, A_n) with $n = 2^2 \cdot 3, 2^3 \cdot 3, 2^2 \cdot 3^2, 2^3 \cdot 3^2, 2^4 \cdot 3^2, 2^4 \cdot 3^3$ or $2^4 \cdot 3^6$. Furthermore, $(G, T) \neq (A_{11}, A_{12})$ by Lemma 3.1. It follows that (G, T) is listed in Table 1. \square

Now, we are ready to prove Theorem 1.1.

The proof of Theorem 1.1: Let G be a non-abelian simple group and Γ a connected tetravalent 2-arc-transitive G -regular graph with $v \in V(\Gamma)$. Then $G_v = 1$ and $|G| = |V(\Gamma)|$. Set $A = \text{Aut}(\Gamma)$ and let R be the radical of A . By Lemma 3.2, the theorem is true for $R = 1$. Thus, we may assume $R \neq 1$.

Set $B = RG$. Then $G \cap R = 1$ and so $|B| = |R||G|$. Since Γ is G -regular, $B = GB_v$ and $|B| = |G||B_v|$. It follows that $|R| = |B_v|$, and by Proposition 2.1, $|R| \mid 2^4 \cdot 3^6$.

Suppose that R has one or two orbits on $V(\Gamma)$. Since Γ is a connected tetravalent G -regular graph, $|R| = |R_v||v^R| = |R_v||G|$ or $|R_v||G|/2$. Since $|R| \mid 2^4 \cdot 3^6$, the non-abelian simple group G is a $\{2, 3\}$ -group, which is impossible.

Thus, R has at least three orbits. If $B = R \times G$ then the theorem is true by Lemma 3.3. Now assume $B \neq R \times G$, and to finish the proof, we aim to derive contradictions.

Since $|R| \mid 2^4 \cdot 3^6$, we may write $|R| = 2^m \cdot 3^k$, where $0 \leq m \leq 4$ and $0 \leq k \leq 6$. Since R is soluble, there exists a series of principle subgroups of B :

$$B > R = R_s > \cdots > R_1 > R_0 = 1,$$

such that $R_i \trianglelefteq B$ and R_{i+1}/R_i is an elementary abelian r -group with $0 \leq i \leq s-1$ and $r = 2$ or 3 . Let $|R_{i+1}/R_i| = r^{\ell_i}$. Then $\ell_i \leq m \leq 4$ if $r = 2$, and $\ell_i \leq k \leq 6$ if $r = 3$. Note that $G \leq B$ has a natural action on R_{i+1}/R_i by conjugation.

Since $B \neq R \times G$, there exists $0 \leq j \leq s-1$ such that $GR_j = G \times R_j$ and $GR_{j+1} \neq G \times R_{j+1}$. If G acts trivially on R_{j+1}/R_j , then $[GR_j/R_j, R_{j+1}/R_j] = 1$. Since $GR_j/R_j \cong G$ is simple, $(GR_j/R_j) \cap (R_{j+1}/R_j) = 1$, and since $|GR_{j+1}/R_j| = |GR_{j+1}/R_{j+1}||R_{j+1}/R_j| = |G||R_{j+1}/R_j| = |GR_j/R_j||R_{j+1}/R_j|$, we have $GR_{j+1}/R_j = GR_j/R_j \times R_{j+1}/R_j$. In particular, $GR_j \trianglelefteq GR_{j+1}$ and so $G \trianglelefteq GR_{j+1}$ because $GR_j = G \times R_j$ implies that G is characteristic in GR_j . It follows that $GR_{j+1} = G \times R_{j+1}$, a contradiction. Thus, G acts non-trivially on R_{j+1}/R_j , and the simplicity of G implies that the action is faithful. Since R_{j+1}/R_j is an

elementary abelian group of order r^{ℓ_j} , we have $G \leq \text{GL}(\ell_j, r)$, where $\ell_j \leq m \leq 4$ if $r = 2$ and $\ell_j \leq k \leq 6$ if $r = 3$.

By Proposition 2.2, R is semiregular on $V(\Gamma)$, and the quotient graph Γ_R is a connected tetravalent $(A/R, 2)$ -arc-transitive graph with $A/R \leq \text{Aut}(\Gamma_R)$. Moreover, Γ_R is B/R -vertex-transitive, and $|B/R| = |V(\Gamma_R)| |(B/R)_\alpha|$ for $\alpha \in V(\Gamma_R)$. Since Γ is G -regular, $|V(\Gamma_R)| = |V(\Gamma)|/|R| = |G|/|R|$, and since $B/R \cong G$, we have $|G| = |B/R| = |V(\Gamma_R)| |(B/R)_\alpha| = |G|/|R| \cdot |(B/R)_\alpha|$. It follows that $|R| \mid |G|$ and $|(B/R)_\alpha| = |R|$.

Since R is the largest normal soluble subgroup of A , A/R has trivial radical, and since $B/R \cong G$, Lemma 3.2 implies that either $B/R \trianglelefteq A/R$, or A/R has a normal arc-transitive subgroup I/R such that $B/R \leq I/R$ and $(B/R, I/R) \cong (G, T) = (M_{11}, M_{12})$ or (A_{n-1}, A_n) with $n \geq 8$ and $n \mid 2^4 \cdot 3^6$.

Case 1: $B/R \trianglelefteq A/R$.

In this case, $B \trianglelefteq A$ and $B_v \trianglelefteq A_v$. Since A is 2-arc-transitive, B is arc-transitive and Γ_R is B/R -arc-transitive. Thus, $4 \mid |B_v|$ and $\Gamma_R \cong \text{Cos}(B/R, (B/R)_\alpha, (B/R)_\alpha g (B/R)_\alpha)$ for some feasible g . Recall that $|R| = |B_v| = |(B/R)_\alpha| = 2^m \cdot 3^k$, $|R| \mid |G|$ and $G \leq \text{GL}(\ell_j, r)$, where $\ell_j \leq m \leq 4$ if $r = 2$ and $\ell_j \leq k \leq 6$ if $r = 3$.

By Proposition 2.1, $A_v \cong A_4, \mathbb{Z}_3 \times A_4, S_4, \mathbb{Z}_3 \rtimes S_4, S_3 \times S_4, \text{AGL}(2, 3)$ or $\mathbb{Z}_3^5 \rtimes \text{GL}(2, 3)$.

Suppose $A_v \cong A_4, \mathbb{Z}_3 \times A_4, S_4, \mathbb{Z}_3 \rtimes S_4$ or $S_3 \times S_4$. Since $|R| = |B_v|$, we have $|R| = 2^m \cdot 3^k$, where $m \leq 4$ and $k \leq 2$. Since $\text{GL}(2, 2)$ and $\text{GL}(2, 3)$ are soluble and $G \leq \text{GL}(\ell_j, r)$, we have either $m = 4$ and $G \leq \text{GL}(4, 2) = \text{PSL}(4, 2)$, or $m = 3$ and $G = \text{PSL}(3, 2)$.

If $m = 4$ and $G \leq \text{PSL}(4, 2)$ then $2^4 \mid |B_v|$ and $A_v = S_3 \times S_4$. Since $B_v \trianglelefteq A_v$, we have $B_v = S_3 \times S_4$, and by Proposition 2.1, $|(B/R)_\alpha| = |B_v| = 2^4 \cdot 3^2$ implies that $(B/R)_\alpha = S_3 \times S_4$. This is impossible because $B/R \cong G \leq \text{PSL}(4, 2)$ and $\text{PSL}(4, 2)$ has no subgroup isomorphic to $S_3 \times S_4$ by MAGMA. If $m = 3$ and $G = \text{PSL}(3, 2)$ then $A_v = S_4, \mathbb{Z}_3 \rtimes S_4$ or $S_3 \times S_4$. Since $B_v \trianglelefteq A_v$ and $2^3 \mid |B_v|$, we have $|(B/R)_\alpha| = |B_v| = 2^3 \cdot 3, 2^3 \cdot 3^2, 2^4 \cdot 3$ or $2^4 \cdot 3^2$, and by Proposition 2.1, $(B/R)_\alpha \cong S_4, \mathbb{Z}_3 \rtimes S_4$ or $S_3 \times S_4$. By Atlas [2, pp.3], $(B/R)_\alpha \cong S_4$ as $B/R \cong \text{PSL}(3, 2)$. By MAGMA, $(B/R)_\alpha$ has two conjugacy classes in B/R , and for both classes, there are no feasible g , a contradiction.

Suppose $A_v \cong \text{AGL}(2, 3)$. Since $4 \mid |B_v|$ and $B_v \triangleleft A_v$, by MAGMA $B_v = A_v$, and so $|R| = 2^4 \cdot 3^3$. Since $3^3 \nmid |\text{GL}(4, 2)|$, we have $G \leq \text{GL}(3, 3)$, and by MAGMA, $G = \text{PSL}(3, 3)$. Since $|(B/R)_\alpha| = |R| = 2^4 \cdot 3^3$, by Proposition 2.1 $(B/R)_\alpha \cong \text{AGL}(2, 3)$. By MAGMA, there are two conjugacy classes isomorphic to $\text{AGL}(2, 3)$ in $\text{PSL}(3, 3)$, and for each conjugacy class, there is no feasible g , a contradiction.

Suppose $A_v \cong \mathbb{Z}_3^5 \rtimes \text{GL}(2, 3)$. Since $4 \mid |B_v|$ and $B_v \trianglelefteq A_v$, by the Remark of Proposition 2.1, $B_v = A_v$ and $|R| = |(B/R)_\alpha| = 2^4 \cdot 3^6$. Thus, B is 7-arc-transitive and $G \cong B/R$ has a subgroup of order $2^4 \cdot 3^6$. Since $2^4 \cdot 3^6 \nmid |\text{GL}(4, 2)|$, we have $G \not\leq \text{GL}(4, 2)$. Thus, $G \leq \text{GL}(\ell_j, 3)$ with $\ell_j \leq k \leq 6$. By MAGMA, for $1 \leq k \leq 4$, $\text{GL}(k, 3)$ have no simple subgroup G with a subgroup of order $2^4 \cdot 3^6$. It follows that $\ell_j = 5$ or 6 with $r = 3$, and hence $|R_{j+1}/R_j| = 3^5$ or 3^6 , where $0 \leq j \leq s-1$.

Let $j \neq s-1$. Then $|R/R_{s-1}| = 3$ or 2^t for $1 \leq t \leq 4$, and so $|R_{s-1}| = 2^4 \cdot 3^5$ or $2^{4-t} \cdot 3^6$. Since G is simple and $G \not\leq \text{GL}(4, 2)$, G acts trivially on R/R_{s-1} by conjugation, implying $[GR_{s-1}/R_{s-1}, R/R_{s-1}] = 1$. It follows that $GR/R_{s-1} = GR_{s-1}/R_{s-1} \times R/R_{s-1}$ and hence $GR_{s-1} \trianglelefteq GR = B$. Since B is 7-arc-transitive, GR_{s-1} is arc-transitive, and since G is

regular, GR_{s-1} has a stabilizer of order $2^4 \cdot 3^5$ or $2^{4-t} \cdot 3^6$, contrary to Proposition 2.1.

Let $j = s - 1$. Then $|R/R_{s-1}| = 3^5$ or 3^6 , and $|R_{s-1}| = 2^4 \cdot 3$ or 2^4 . Furthermore, $GR_{s-1} = G \times R_{s-1}$ and $B = GR \neq G \times R$. Clearly, $R = R_{s-1}P$ for a Sylow 3-subgroup P of R . Let $C = C_B(R_{s-1})$. Then $C \trianglelefteq B$ and $G \leq C$. If $G = C$ then $G \triangleleft B$ and so $B = G \times R$, a contradiction. Thus, G is a proper subgroup of C , so that $C_v \neq 1$. Since B is 7-arc-transitive, C is arc-transitive and by the Remark of Proposition 2.1, $C_v = B_v$, that is, $B = C$. In particular, $[P, R_{s-1}] = 1$. Since $R = PR_{s-1}$, P is normal in R and so characteristic. This implies $P \trianglelefteq B$ and so $GP \leq B = C$. It follows $[GP, R_{s-1}] = 1$, and since $B = GR = (GP)R_{s-1}$, we have $GP \trianglelefteq B$. Since B is 7-arc-transitive, GP is arc-transitive, forcing $4 \mid |(GP)_v|$, and since G is regular, $|(GP)_v| = |P|$ and thus $4 \mid |P|$, contrary to the fact that P is a Sylow 3-subgroup of R .

Case 2: A/R has a normal arc-transitive subgroup I/R such that $B/R \leq I/R$ and $(B/R, I/R) \cong (G, T) = (M_{11}, M_{12})$ or (A_{n-1}, A_n) with $n \geq 8$ and $n \mid 2^4 \cdot 3^6$.

In this case, $I \trianglelefteq A$ and I is arc-transitive. Since $|T| = |I/R| = |V(\Gamma_R)| |(I/R)_\alpha| = |G|/|R| \cdot |(I/R)_\alpha|$, we have $|(I/R)_\alpha| = |R||T|/|G|$. By Proposition 2.1, $|R||T|/|G|$ is a divisor of $2^4 \cdot 3^6$. Recall that $|R| = |B_v| = |(B/R)_\alpha| = 2^m \cdot 3^k$, $|R| \mid |G|$ and $G \leq \text{GL}(\ell_j, r)$, where $\ell_j \leq m \leq 4$ if $r = 2$ or $\ell_j \leq k \leq 6$ if $r = 3$.

Suppose $(G, T) = (M_{11}, M_{12})$. Then $|T|/|G| = 2^2 \cdot 3$. Since $|R||T|/|G| \mid 2^4 \cdot 3^6$, we have $|R| \mid 2^2 \cdot 3^5$, and then $|R| \mid |G|$ implies that $|R| \mid 2^2 \cdot 3^3$. Since $\text{GL}(2, 2)$ is soluble, we have $G \leq \text{GL}(3, 3)$, which is impossible because $11 \mid |M_{11}|$ and $11 \nmid |\text{GL}(3, 3)|$.

Suppose $(G, T) = (A_{n-1}, A_n)$ with $n \geq 8$ and $n \mid 2^4 \cdot 3^6$. If $n \geq 12$ then $5^2 \mid |G|$, which is impossible because $G \leq \text{GL}(4, 2)$ or $\text{GL}(6, 3)$ but $5^2 \nmid |\text{GL}(6, 3)|$ and $5^2 \nmid |\text{GL}(4, 2)|$. It follows that $8 \leq n < 12$, and since $n \mid 2^4 \cdot 3^6$, we have $(G, T) = (A_7, A_8)$ or (A_8, A_9) .

Suppose $(G, T) = (A_7, A_8)$. Then $|T|/|G| = 2^3$. Since $|R||T|/|G| \mid 2^4 \cdot 3^6$, we have $|R| \mid 2 \cdot 3^6$, and $|R| \mid |G|$ implies that $|R| \mid 2 \cdot 3^2$. Thus, $G \leq \text{GL}(1, 2)$ or $\text{GL}(2, 3)$, yielding that G is soluble, a contradiction.

Suppose $(G, T) = (A_8, A_9)$. Then $|T|/|G| = 3^2$. Since $|R||T|/|G| \mid 2^4 \cdot 3^6$, we have $|R| \mid 2^4 \cdot 3^4$ and $|R| \mid |G|$ implies $|R| \mid 2^4 \cdot 3^2$, that is, $|R| = 2^m \cdot 3^k$ with $m \leq 4$ and $k \leq 2$. Since $\text{GL}(2, 3)$ is soluble, $G \leq \text{GL}(4, 2)$, and hence $G = \text{GL}(4, 2)$ and $m = 4$ as $G = A_8 \cong \text{GL}(4, 2)$. Since $|(I/R)_\alpha| = |R||T|/|G|$, we have $2^4 \cdot 3^2 \mid |(I/R)_\alpha|$, and since $|I/R| = |A_9| = 2^6 \cdot 3^4 \cdot 5 \cdot 7$, Proposition 2.1 implies that $(I/R)_\alpha \cong S_3 \times S_4$ or $\text{AGL}(2, 3)$. By the arc-transitivity of I , Γ_R is I/R -arc-transitive, and so $\Gamma_R = \text{Cos}(I/R, (I/R)_\alpha, (I/R)_\alpha t (I/R)_\alpha)$ for some feasible t . By MAGMA, $I/R \cong A_9$ has no subgroup isomorphic to $\text{AGL}(2, 3)$ and I/R has two conjugacy classes isomorphic to $S_3 \times S_4$. By taking a given $(I/R)_\alpha$ in each conjugacy class, computation shows that there is no feasible t , a contradiction. \square

To end the paper, we give examples to show that the pair $(G, T) = (A_{23}, A_{24})$ in Theorem 1.1 can happen.

Example 3.4 Let $G = A_{23}$ and $T = A_{24}$. Define $x, y, z, w, g \in T$ as following:

$$\begin{aligned} x &= (1, 2)(3, 7)(4, 10)(5, 13)(6, 15)(8, 12)(9, 19)(11, 18)(14, 22)(16, 20)(17, 24)(21, 23), \\ y &= (1, 3)(2, 7)(4, 8)(5, 9)(6, 18)(10, 12)(11, 15)(13, 19)(14, 20)(16, 22)(17, 23)(21, 24), \end{aligned}$$

$$\begin{aligned}
z &= (1, 4, 6)(2, 8, 11)(3, 12, 15)(5, 17, 16)(7, 10, 18)(9, 21, 20)(13, 23, 14)(19, 24, 22), \\
w &= (1, 5)(2, 9)(3, 13)(4, 16)(6, 17)(7, 19)(8, 20)(10, 22)(11, 21)(12, 14)(15, 23)(18, 24), \\
g_1 &= (1, 5)(2, 10)(3, 14)(4, 17)(6, 16)(7, 11)(8, 18)(9, 22)(12, 13)(15, 23)(19, 21)(20, 24), \\
g_2 &= (1, 5)(2, 10)(3, 8)(4, 16)(6, 15)(7, 19)(9, 22)(11, 12)(13, 20)(14, 18)(17, 24)(21, 23), \\
g_3 &= (1, 5)(2, 9)(3, 13)(4, 16)(6, 15)(7, 14)(8, 20)(10, 19)(11, 17)(12, 23)(18, 22)(21, 24), \\
g_4 &= (1, 2)(3, 12)(4, 8)(5, 9)(6, 10)(7, 19)(11, 23)(13, 22)(14, 18)(15, 20)(16, 21)(17, 24).
\end{aligned}$$

By MAGAMA[1], $H = \langle x, y, z, w \rangle \cong S_4$, $T = \langle H, g_i \rangle$, $|H : H \cap H^{g_i}| = 4$ ($1 \leq i \leq 4$) and H is regular on $\{1, 2, \dots, 24\}$. Thus, $T = GH$ with $G \cap H = 1$. Define $\Gamma_i = \text{Cos}(T, H, Hg_iH)$ with $1 \leq i \leq 4$. Then Γ_i is a connected tetravalent $(T, 2)$ -arc-transitive G -regular graph, where G and T are viewed as groups of automorphisms of Γ_i by right multiplication. By Theorem 1.1, $T \trianglelefteq \text{Aut}(\Gamma_i)$ with $1 \leq i \leq 4$. Again by MAGMA, $\text{Aut}(T, H, Hg_iH) \cong \tilde{H}$, where \tilde{H} is the automorphism group of T induced by conjugate of elements in H . Thus, $\text{Aut}(\Gamma_i) = T$ ($1 \leq i \leq 4$) by [18, Lemma 2.10].

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Appendix

Let G be a finite group and $H \leq G$. Let $\Gamma = \text{Cos}(G, H, HgH)$ be a connected tetravalent G -arc-transitive graph. Then g can be chosen as a 2-element such that $g^2 \in H$, $\langle H, g \rangle = G$, $|H : H \cap H^g| = 4$, and such a g is called feasible. There are many places in the paper to compute feasible g for given G and H and to compute their full automorphism groups of the corresponding coset graphs. Here we provide a computer program based on MAGMA language by taking $(G, H) = (M_{12}, A_4)$ as an example:

```
load "M12"; B:=Subgroups(G);
PG:=[]; //possible graphs
for i in [1..#B] do if IsIsomorphic(Alt(4),B[i]'subgroup) then H:=B[i]'subgroup; print i;
D:=[]; // feasible elements
for g in G do if IsDivisibleBy (2^6, Order(g)) and Order(sub <G|H, g^2>) eq Order(H) and
#(H * g * H) eq Order(H)*4 and Order(sub <G|H, g>) eq Order(G) then Include (~ D, g);
end if; end for;
#D;
if #D ne 0 then for j in [1..#D] do c:=D[j]; HcH:={};
for t in H do for h in H do Include (~ HcH, t * c * h); end for; end for;
Vj:={}; Ej:={};
for t in G do for s in HcH do T1:={}; T2:={}; for h in H do Include (~ T1, h * t); Include
(~ T2, h * s * t); end for; Include (~ Vj, T1); Include (~ Ej, {T1, T2}); end for; end for;
PGj:=Graph <Vj|Ej>; Include(~ PG, PGj);
end for; end if; end if; end for;
NPG:=[]; // non-isomorphic possible graphs
NPG:=[PG[1]];
for k in [1..#PG] do p:=0; for m in [1..#NPG] do if IsIsomorphic(PG[k], NPG[m]) then
p:=p+0; else p:=p+1; end if; end for;
if p eq #NPG then Include(~ NPG, PG[k]); end if; end for;
Graph:=[]; //M12-arc-transitive M11-regular graph
for t in [1..#NPG] do A:=AutomorphismGroup (NPG[t]); S:=Subgroups(A);
for n in [1..#S] do if IsTransitive(S[n]'subgroup) and Order(S[n]'subgroup) eq 7920 and
IsSimple(S[n]'subgroup) then Include(~ Graph, NPG[t]);
print "We find a connected tetravalent M12-arc-transitive M11-regular graph";
print "The automorphism group A:"; print A;
print "The radical of A:"; print Radical(A);
end if; end for; end for;
#Graph;
```

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